II. *On the Method of determining, from the real Probabilities of Life, the Values of Contingent Reversions in which three Lives are involved in the Survivorship.* By William Morgan, *Esq. F. R. S.*

Read December 12, 1799.

ON SURVIVORSHIPS.

THE several papers which I have had the honour of communicating to the Royal Society, on the doctrine of contingent reversions, contain the greater number of those cases in which three lives are concerned in the survivorship. With the view of completing this subject, I have been induced to investigate the remaining problems; and, having succeeded in the solution of them, I hope the following will not be considered as an improper addition to my former communications.

Being anxious to render this paper as concise as possible, I have omitted to state at length the different contingencies on which the payment of the given sum depends; trusting that, from an attentive perusal of my former demonstrations, these will appear to be so plainly expressed by the several fractions in each problem, as to render a more ample description of them unnecessary.

PROBLEM I.

To determine the value of a given sum, payable on the death of A or B, should either of them be the *first* or *second* that fails, of the three lives, A, B, and C.

Solution.

In this case, the payment of the given sum must certainly take place on the extinction of the joint lives of A and B, independent of C, and therefore the value of the reversion will be $= \frac{S.\overline{r-1}.\overline{V-AB}}{r}$. *

The fractions expressing the contingencies on which the payment of S depends, in the 1st year, are $\frac{S}{abcr} \times \overline{a'.\overline{b-m}.\overline{c-d} + a'.\overline{b-m}.d + a'.\overline{c-d}.m + \overline{b-m}.\overline{c-d}.\overline{a-a'}} = \frac{S}{abcr} \times \overline{abc - mc.\overline{a-a'}}$; in the 2d year $= \frac{S}{abcr^2} \times \overline{a''.\overline{m-n}.\overline{d-e} + a''.\overline{m-n}.e + a'' \times \overline{d-e}.n + a''.ne + \overline{m-n}.\overline{d-e}.\overline{a-a'+a''} + \overline{m-n}.\overline{a-a'+a''}.e + \overline{c-d}.n.a'' + \overline{c-d}.\overline{m-n}\,\overline{a-a'+a''} + \overline{c-d}.\overline{m-n}.a''} = \frac{S}{abcr^2} \times \overline{mc.\overline{a-a} - nc.\overline{a-a'+a''}}$, and so on in the other years; hence the whole value is $= \frac{S}{r} \times \overline{\overline{r-1}.V - r.AB + AB} = \frac{S.\overline{r-1}}{r}.\overline{V-AB}$, as before. Q.E.D.

PROBLEM II.

To determine the value of a given sum, payable on the decease of A or B, should either of them be the *second* or *third* that shall fail, of the three lives, A, B, and C.

* The same symbols are uniformly retained in this, as in my last two papers on the subject. See Phil. Trans. Vol. LXXXI. page 247.

Solution.

In the 1st year, the value of the reversion will be $\frac{S}{abcr}$ into $a'.\overline{b-m}.\overline{c-d}+a.\overline{b-m}.d+\frac{a.\overline{c-d}.m}{2}+\frac{\overline{b-m}.\overline{c-d}.\overline{a-a'}}{2}$; in the 2d year it will be $\frac{S}{abcr^2}$ into $a''.\overline{m-n}.\overline{d-e}+a''.\overline{m-n}.e+\frac{a''.\overline{d-e}.n}{2}+\frac{\overline{m-n}.\overline{d-e}.\overline{a-\overline{a'+a''}}}{2}+a'.\overline{m-n}.\overline{d-e}+\overline{b-m}.\overline{d-e}.a''+\overline{c-d}.\overline{m-n}.a''+a'.\overline{m-n}.e+\overline{b-m}.a''.e+\overline{c-d}.\overline{m-n}.\overline{a-\overline{a'+a''}}+\overline{c-d}.n.a''+\frac{a'.\overline{c-d}.\overline{m-n}}{2}+\frac{\overline{b-m}.\overline{c-d}.a''}{2}$; in the 3d year it will be $\frac{S}{abcr^3}$ into $a'''.\overline{n-o}.\overline{e-f}+a''.\overline{n-o}.f+\frac{a'''.\overline{e-f}.o}{2}+\overline{n-o}.\overline{e-f}.\overline{a'+a''}+\frac{\overline{n-o}.\overline{e-f}.\overline{a-\overline{a'+a''+a'''}}}{2}+\overline{b-n}.\overline{e-f}.a'''+\overline{c-e}.\overline{n-o}.a'''+\overline{n-o}.f.\overline{a'+a''}+\overline{b-n}.f.a'''+\overline{c-e}.\overline{n-o}.\overline{a-\overline{a'+a''+a'''}}+\overline{c-e}.o.a'''+\frac{\overline{c-e}.\overline{n-o}.\overline{a'+a''}}{2}+\frac{\overline{b-n}.\overline{c-e}.a'''}{2}$, and so on. These different fractions being added together, will be found

$$=\frac{S}{b}\times\overline{\frac{b-m}{r}+\frac{m-n}{r^2}+\frac{n-o}{r^3},\ \&c.}-\frac{S}{2bc}\times\overline{\frac{\overline{b-m}.c}{r}+\frac{\overline{m-n}.d}{r^2}+\frac{\overline{n-o}.e}{r^3},\ \&c.}-\frac{S}{2bc}\times\overline{\frac{\overline{b-m}.d}{r}+\frac{\overline{m-n}.e}{r^2}+\frac{\overline{n-o}.f}{r^3}+,\ \&c.}+$$

$$\frac{S}{2a}\times\overline{\frac{a'}{r}+\frac{a''}{r^2}+\frac{a'''}{r^3}+,\ \&c.}+\frac{S}{2ac}\times\overline{\frac{ca'}{r}+\frac{da''}{r^2}+\frac{ea'''}{r^3}+,\&c.}+\frac{S}{2ab}\times\overline{\frac{ba'}{r}+\frac{ma''}{r^2}+\frac{na'''}{r^3}+,\ \&c.}-\frac{S}{2abc}\times\overline{\frac{bca'}{r}+\frac{mda''}{r^2}+\frac{nea'''}{r^3}+,\ \&c.}$$

$$-\frac{S}{2abc}\times\overline{\frac{mca'}{r}+\frac{nda''}{r^2}+\frac{oea'''}{r^3}+,\ \&c.}-\frac{S}{2abc}\times\overline{\frac{mdd'}{r}+\frac{nea''}{r^2}+\frac{ofa'''}{r^3}+,\ \&c.}\quad+\frac{S}{ab}\times\overline{\frac{ma'}{r}+\frac{n.\overline{a'+a''}}{r^2}+}$$

$\frac{o\cdot\overline{a'+a''+a'''}}{r^3}+$, &c. $-\frac{S}{ab}\times\overline{\frac{ba'}{r}+\frac{m\cdot\overline{a'+a''}}{r^2}+}$, &c. $+\frac{S}{2abc}$ $\times\overline{\frac{bc\cdot a'}{r}+\frac{md\cdot\overline{a'+a''}}{r^2}+}$, &c. $-\frac{S}{2abc}\times\overline{\frac{mca'}{r}+\frac{nd\cdot\overline{a'+a''}}{r^2}+}$, &c. $+\frac{S}{2abc}\times\overline{\frac{bda'}{r}+\frac{me\cdot\overline{a'+a''}}{r^2}+}$, &c. $-\frac{S}{2abc}\times\overline{\frac{md\cdot a'}{r}+\frac{ne\cdot\overline{a'+a''}}{r^2}+}$, &c. $+\frac{S}{2abr}\times\overline{\frac{ma'}{r}+\frac{n\cdot\overline{a'+a''}}{r^2}}$, &c. $-\frac{S}{2abr}\times\overline{\frac{n\cdot d}{r}+\frac{o\cdot\overline{a'+a''}}{r^2}+}$, &c. $+\frac{S}{2abcr}\times\overline{\frac{md\cdot a'}{r}+\frac{ne\cdot\overline{a'+a''}}{r^2}+}$, &c. $-\frac{S}{2abcr}\times\overline{\frac{nd\cdot a'}{r}+\frac{oe\cdot\overline{a'+a''}}{r^2}}$, &c. which may at last be reduced to S into $\frac{r-1}{r}\times\overline{V+ABC-\frac{A+B+C}{2}+\frac{AC}{2r}-AB}-\frac{\varkappa}{2c}\times\overline{AK+BK-2ABK}+\frac{\beta}{2b}\times\overline{AF-AFC}+\frac{m}{2br}\times\overline{1+AP}+\frac{d\cdot\overline{PT-APT}}{c}$. But, unless A and B are very nearly of the same age, and both older than C, this rule will not be sufficiently accurate. If *B be the oldest of the three lives,* the annuities A, AC, and AK, should be continued only for as many years (x) as are equal to the difference between the age of B and that of the oldest life in the table of observations. Let those annuities be respectively denoted by A′, A′C′, and A′K′; also let φ denote the probability that C survives B,* q the number of persons living opposite to the age of A at the end of x years, then will the value of S, after x years, be $=\frac{\varphi q}{ar^x}\times\frac{\overline{r-1}\cdot\overline{V-A^x}}{r}$, and the whole value of the reversion will be $=$ S into $\frac{\overline{r-1}}{r}\times\overline{V+ABC-\frac{A'+B+BC}{2}\times\frac{A'C'}{2r}-AB}-\frac{\varkappa}{2c}\times\overline{A'K'+BK-2BK}+\frac{\beta}{2b}\times\overline{AF-AFC}+\frac{m}{2br}\times\overline{1+AP}$

* By the Table, page 337, Phil. Trans. Vol. LXXVIII.

$+\frac{d\,.\,\overline{PT-APT}}{c}+\frac{\phi q}{a r^{x}}\times\frac{\overline{r-1}\,.\,\overline{V-A^{x}}}{r}$, ($A^x$ denoting the value of an annuity on a life x years older than A).

If C *be the oldest of the three lives,* let $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, *&c.* be substituted for their equals a', a'', a''', *&c.* and $\overline{c-c'}$, $\overline{c-\overline{c'+c''}}$, $\overline{c-\overline{c'+\overline{c''+c'''}}}$, *&c.* for their equals e, f, g. *&c.* then will the value of the reversion, by pursuing the same steps as in the former case, be found $=$ S into $\frac{r-1}{r}\times$ $\overline{V-\frac{1}{2}\overline{A+B}+ABC}+\frac{BC+AC}{2r}-AB+\frac{\beta\,.\,\overline{AF-AFC}}{2b}+\frac{\alpha\,.\,\overline{HB-HBC}}{2a}-\frac{s}{2ar}\times\overline{1+NC}-\frac{m}{2br}\times\overline{1+PC}+\frac{ms}{abr}\times\overline{1+NPC}$.

But, as these series are to be continued only during C's life, it is evident that the annuities A, B, AB, *&c.* should also be continued only during this term; and therefore, if z be the difference between the age of C and the oldest person in the Table, A′, B′, A′B′, *&c.* the values of annuities respectively on the single and joint lives of A, B, A and B, *&c.* for z years, these several symbols should be substituted above, in lieu of A, B, AB, *&c.* to denote the value of the reversion during the first z years. After the extinction of the life of C, the given sum may be received upon either of three events; 1st, if A should have died before C, and B died after him in the $\overline{z+1}$, $\overline{z+2}$, *&c.* years; 2dly, if B should have died before C, and A died after him in those years respectively; 3dly, if both the lives of A and B should die after the first z years. Let ϕ denote the probability that C dies after A, and π the probability that C dies after B,* then will the value of the reversion depending on these several contingencies (putting p for the number of persons living opposite the age of B at the end of z years, and q for the same number opposite the age of A) be $= S$ into $\frac{p \,.\, \phi \,.\, \overline{r-1} \,.\, \overline{V-B^z}}{b\,r^{z+1}} + \frac{q\pi \,.\, \overline{r-1} \,.\, \overline{V-A^z}}{a\,r^{z+1}} + \frac{pq}{a\,b\,r^z} \times \overline{\frac{1+AB^z}{r} - AB^z}$, and the whole value of the reversion will be $= S$ into

$$\frac{\overline{r-1} \,.\, \overline{V - \frac{A'+B}{2} + ABC}}{r} + \frac{BC+AC}{2r} - AB + \frac{\beta \,.\, \overline{A'F'-AFC}}{2b} + \frac{\alpha \,.\, \overline{H'B'-HBC}}{2a} - \frac{s}{2ar} \times \overline{1+NC} + \frac{m}{br} \times \overline{\frac{s \,.\, \overline{1+PNC}}{a} - \frac{\overline{1+PC}}{2}} + \frac{p \,.\, \phi \,.\, \overline{r-1} \,.\, \overline{V-B^z}}{b\,r^{z+1}} + \frac{\pi \,.\, q \,.\, \overline{r-1} \,.\, \overline{V-A^z}}{a\,r^{z+1}} + \frac{pq \,.}{a\,b\,r^{z+1}} \times \overline{1+AB^z}.\dagger$$

If the lives be all equal, the value, according to the first rule, will be $= S$ into $\frac{r-1}{r} \times \overline{V - C - \frac{1}{2}CC + CCC} + \frac{CC}{2r} - CC + \frac{\varkappa}{2c} \times \overline{KCC - KC} + \frac{d}{2cr} \times \overline{1+CT} + \frac{dd}{2ccr} \times \overline{TT - CTT}$; and, according to the 2d rule, it will be $= S$ into $\frac{r-1}{r} \times \overline{V - C + CCC} + \frac{CC}{r} - CC + \frac{\varkappa \,.\, \overline{CK-CCK}}{c} - \frac{d}{cr} \times \overline{1+CT} + \frac{dd}{ccr} \times \overline{1+CTT}$. If these expressions be resolved into their respective series, the value in each case will be found $= \frac{S \,.\, \overline{r-1}}{r} \times \overline{V - C - CC + CCC}$, which is known to be the true value, from self-evident principles.

But the solution of this problem may be obtained by the

* By the Table, page 229, Phil. Trans. for the year 1794.

† In this and the following problems, A^z, B^z, AB^z, A^x, B^x, &c. signify the value of an annuity on the single or joint lives of persons z or x years older than A and B, &c.

assistance of the 1st problem in my last paper,* supposing, instead of a *given sum*, it were required to know the value of the reversion of a given *estate.* For, since the possession of this estate is an event which must certainly take place, and the only point to be determined is the *time* in which it will probably happen, it is obvious that no event can postpone the possession, but the contingency of C's being the *second* that fails, of the three lives. If, therefore, the sum of the values of an annuity on the life of B after A, provided A should die before C, and of an annuity on the life of A after B, provided B should die before C, (both found by the problem just mentioned,) be subtracted from the whole value of the reversion after the joint lives of A and B, the remainder will be the value required. Let X and Y respectively denote the annuities found by problem 1st, (Phil. Trans. Vol. LXXXIV.) then will the general rule expressing the value of an *estate* be $= V - AB - \overline{X + Y}$, and consequently of a given *sum* $= \frac{S . \overline{r-1}}{r} \times \overline{V - AB - \overline{X + Y}}$, which, when the lives are equal, may be reduced, as in the former cases, to $\frac{S . \overline{r-1}}{r} \times \overline{V - C - CC + CCC}$. *Q. E. D.*

PROBLEM III.

To determine the value of an *estate*, or of a given *sum*, after the decease of A or B, should either of them be the *first* or *last* that shall fail, of the three lives, A, B, and C.

Solution.

The reversion of the *estate* in this problem, like that in the

* Phil. Trans. for the year 1794, page 235.

preceding one, cannot be prevented ultimately from taking place; and there is only the single contingency of C's being the *first* that fails, of the three lives, which can postpone the possession of it after the extinction of the joint lives. The whole value, therefore, of the reversion, after the joint lives of A and B, must in this case be lessened by the *sum* of the values of an annuity on A's life after B, provided B should survive C; and of an annuity on B's life after A, provided A should survive C, (both found by the 2d problem in my last Paper.*) Let these two values be respectively denoted by W and Z, then will the general rule expressing the value of an *estate* be $= V - AB - \overline{W + Z}$, and the value of a given *sum* $= \frac{S.\overline{r-1}}{r} \times \overline{V - AB - W + Z}$. When the lives are all *equal*, the value of the reversion, by substituting the values of W and Z, becomes $= V + CC - C - CCC$, or $\frac{S.\overline{r-1}}{r} \times \overline{V + CC - C - CCC}$, according as it consists of an *estate*, or a given *sum*.

But the solution of this, like that of the preceding problem, may be obtained without having recourse to any other. In the first year, the value of the given sum will be = S into $\frac{a'.\overline{b-m}.\overline{c-d}.}{abcr} + \frac{a'.\overline{b-m}.d}{abcr} + \frac{a'.\overline{c-d}.m}{2abcr} + \frac{\overline{b-m}.\overline{c-d}.\overline{a-a'}}{2abcr} + \frac{a'md}{abcr} + \frac{\overline{b-m}.\overline{a-a'}.d}{abcr}$; in the 2d year it will be $= \frac{S}{abcr^2}$ into $\overline{m-n}.\overline{d-e}.a'' + \overline{m-n}.e.a'' + \frac{\overline{d-e}.n.a''}{2} + \frac{\overline{m-n}.\overline{d-e}.\overline{a-a'+a''}}{2} + a''ne + \overline{m-n}.e.\overline{a-a'+a''} + \overline{c-d}.\overline{m-n}.a'' + \frac{\overline{c-d}.\overline{m-n}.a'}{2} + \frac{\overline{b-m}.\overline{c-d}.a'}{2}$; in the 3d year $= \frac{S}{abcr^3}$ into

* Phil. Trans. for the year 1794, page 240.

$\overline{n-o}\,.\,\overline{e-f}\,.\,a''' + \overline{n-o}\,.\,f a''' + \frac{\overline{e-f}\,.\,o\,.\,a'''}{2} + \frac{\overline{n-o}\,.\,\overline{e-f}\,.\,\overline{a-\overline{a'+a''+a'''}}}{2}$ $+ a'''\,.\,of + \overline{n-o}\,.\,f\,.\,\overline{a-\overline{a'+a''+a'''}} + \overline{c-e}\,.\,\overline{n-o}\,.\,a''' + \frac{\overline{a'+a''}\,.\,\overline{c-e}\,.\,\overline{n-o}}{2} + \frac{\overline{b-n}\,.\,\overline{c-e}\,.\,a'''}{2}$, and so on in the other years. If these several fractions be expanded, they will form nineteen different series, whose sum may be found $=$ S into $\frac{r-1}{r} \times \overline{V - ABC - \frac{A+B}{2} - \frac{AC+BC}{2r} + AB + \frac{\varkappa}{2c} \times \overline{AK+BK-2ABK} - \frac{\beta\,.\,\overline{AF-AFC}}{2b} + \frac{m}{2br} \times \overline{\frac{d\,.\,\overline{1+APT}}{c} - \overline{1+AP}}}.$

If B be the oldest of the three lives, let π denote the probability that B dies *after* C,* then will the value of S, after the extinction of the life of B, be $= \frac{S\,.\,\pi\,.\,q}{a r^x} \times \frac{\overline{r-1}\,.\,\overline{V-A^x}}{r}$, and the whole value of the reversion will be $=$ S into $\frac{r-1}{r} \times \overline{V - ABC - \frac{A'+B}{2} - \frac{A'C'+BC}{2r} + AB - \frac{\beta\,.\,\overline{AF-AFC}}{2b} + \frac{\varkappa}{2c} \times \overline{A'K' + BK - 2ABK} + \frac{m}{2br} \times \overline{\frac{d\,.\,\overline{1+APT}}{c} - \overline{1+AP}} + \frac{\pi\,.\,q}{a r^x} \times \overline{r-1}\,.\,\overline{V-A^x}}$ .... q, A′, A′C′, &c. denoting the same quantities as in the former part of the preceding problem.

If C be the oldest of the three lives, let the symbols be changed as in the preceding problem, and the value of the given sum will be $=$ S into $\frac{r-1}{r} \times \overline{V - ABC - \frac{1}{2}\overline{A'+B'} - \frac{\beta\,.\,\overline{A'F'-AFC}}{2b} - \frac{\alpha\,.\,\overline{H'B'-HBC}}{2a} - \frac{ms}{abr} \times \overline{1+NPC} - \frac{AC+BC}{2r} + A'B' + \frac{m}{2br} \times \overline{1+PC} + \frac{s}{2ar} \times \overline{1+NC}}$.... After the decease of C, the

* By the Table, page 229, Phil. Trans. for the year 1794.

given sum may be received, provided either of three events shall happen; 1st, if A shall have died *after* C in the first z* years, and B dies in the $\overline{z+1}\,.\,\overline{z+2}$, *&c.* year; 2dly, if B having died *after* C in the first z years, A dies in the $\overline{z+1}\,.\,\overline{z+2}$, *&c.* year; 3dly, if both A and B having survived the first z years, the survivor of them dies in any of the following years. Let π denote the probability that A dies after C, and φ the probability that B dies after C, (both found by the Table in Phil. Trans. Vol. LXXXIV. page 229) and let all the other symbols be the same as in the latter part of the preceding problem, then will the value of S, after z years, be $=$ S into $\frac{\pi\,.\,q\,.\,\overline{r-1}}{a\,r^{z+1}} \times \overline{V - A^z} + \frac{\varphi\,.\,p\,.\,\overline{r-1}}{b\,r^{z+1}} \times \overline{V - B^z} + \frac{p\,.\,q}{a\,b\,r^{z+1}} \times \overline{1 - AB^z + \overline{A^z + B^z}} - \frac{p\,q\,.}{a\,b\,r^z} \times \overline{\overline{A^z + B^z} - AB^z}$, and the whole value of the reversion will be $=$ S into $\frac{r-1}{r} \times \overline{V - ABC - \frac{1}{2}\overline{A' + B'}} - \frac{\beta\,.\,\overline{A'F' - AFC}}{2\,b} - \frac{\alpha\,.\,\overline{H'F' - HFC}}{2\,a} + \frac{m}{b\,r} \times \overline{\frac{1+PC}{2} - \frac{s\,.\,\overline{1+NPC}}{a}} - \frac{AC+BC}{2\,r} + AB + \frac{s}{2\,a\,r} \times \overline{1 + NC} + \frac{\pi\,.\,q\,.\,\overline{r-1}}{a\,r^{z+1}} \times \overline{V - A^z} + \frac{\varphi\,p\,.\,\overline{r-1}}{b\,r^{z+1}} \times \overline{V - B^z} + \frac{p\,.\,q\,.\,\overline{r-1}}{a\,b\,r^{z+1}} \times \overline{V - \overline{A^z + B^z}} - \frac{p\,q}{a\,b\,r^{z+1}} \times AB^z$......

If the three lives be equal, the first of these rules becomes $=$ S into $\frac{r-1}{r} \times \overline{V - C + CC - CCC} + \frac{x}{2\,c} \times \overline{CK - CCK} + \frac{d\,d}{2\,c\,c\,r} \times \overline{1 + CTT} - \frac{d}{2\,c\,r} \times \overline{1 + CT}$; and the second $=$ S into $\frac{r-1}{r} \times \overline{V - C + CC - CCC} - \frac{x}{c} \times \overline{CK - CCK} - \frac{d\,d}{c\,c\,r} \times \overline{1 + CTT} + \frac{d}{c\,r} \times \overline{1 + CT}$; but the three last fractions in

* z being, as usual, the number of years between the age of C and the age of the oldest person in the table.

each of those rules destroy one another, and therefore, in both of them the value of the reversion is $= \frac{S \cdot \overline{r-1}}{r} \times \overline{V - C + CC - CCC}$ .... *Q. E. D.*

PROBLEM IV.

To determine the value of a given sum, payable on the death of A, should his life be the *first* or *second* that fails, and should B's life, if it fail, become extinct before the life of C.

Solution.

In this case, the payment of the given sum can only be prevented by the contingency of C's dying before A, and therefore its value is immediately found by the solution of the 2d problem in my first Paper on this subject,* being no more than " the Value of a given Sum on the Death of A, should C sur- " vive him."

The accuracy of this solution will appear from the following investigation. In the 1st year, the given sum will become payable should either of four events take place, the probabilities of which are expressed by the four fractions $\frac{a' \cdot \overline{b-m} \cdot \overline{c-d}}{2abc} + \frac{a' \cdot \overline{b-m} \cdot d}{abc} + \frac{a'md}{abc} + \frac{a' \cdot \overline{c-d} \cdot m}{2abc} = \frac{a'c}{2ac} + \frac{a'd}{2ac}$. In the 2d year, it will become payable provided either of six events should take place, expressed by the fractions $\frac{a'' \cdot \overline{m-n} \cdot \overline{d-e}}{2abc} + \frac{a'' \cdot \overline{m-n} \cdot e}{abc} + \frac{a''ne}{abc} + \frac{a'' \cdot \overline{d-e} \cdot n}{2abc} + \frac{\overline{b-m} \cdot \overline{d-e} \cdot a''}{2abc} + \frac{\overline{b-m} \cdot e \cdot a''}{abc} = \frac{a''d}{2ac} + \frac{a''e}{2ac}$. In the third year, the payment of the given sum will depend on the

* Phil. Trans. Vol. LXXVIII. page 341.

same number of events, and the probabilities of the several contingencies may be reduced to $\frac{a'''e}{2ac} + \frac{a'''f}{2ac}$. The value, therefore, of the reversion will be $= \frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''.d}{r^2} + \frac{a'''e}{r^3}, \&c.}$ $+ \frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3}}$, &c. which two series are known to express the value of S, on the contingency of C's surviving A.*

Were a further proof necessary, it might be observed, that the value of the given sum, in this problem, is equal to the sum of the values of the two reversions depending on the contingency of A's being the *first* that shall fail, of the three lives; and on the contingency of C's surviving A, in case B shall be then dead. Supposing *the three lives to be equal*, these values will be $= \frac{S.\overline{r-1}}{6r} \times \overline{V - 3CC - 2CCC} + \frac{S.\overline{r-1}}{3r} \times \overline{V - CCC}$† $= \frac{S.\overline{r-1}}{2r} \times \overline{V - CC}$, or the value of the reversion depending on one life's surviving the other. . . . . . *Q. E. D.*

PROBLEM V.

To determine the value of a given sum, payable on the death of A, should his life be the *second* or *third* that fails, and should B's life, when it fails, become extinct before the life of C.

Solution.

The value of the given sum, in the 1st year, will be $= \frac{S}{abcr}$ into $\frac{a'.\overline{b-m}.\overline{c-d}}{2} + \frac{a'.\overline{b-m}.d}{2}$; in the 2d year, it will be =

* See Phil. Trans. Vol. LXXVIII. page 342.

† See Phil. Trans. Vol. LXXIX. page 49; and Vol. LXXXI. page 253.

$\frac{S}{abcr^2}$ into $\frac{a''.\overline{m-n}.\overline{d-e}}{3} + \frac{a''.\overline{m-n}.e}{2} + \overline{b-m}.\overline{d-e}.a'' +$
$\overline{b-m}.e.a'' + \frac{\overline{b-m}.\overline{c-d}.a''}{2}$; in the 3d year $= \frac{S}{abcr^3}$ into
$\frac{a'''.\overline{n-o}.\overline{e-f}}{3} + \frac{a'''.\overline{n-o}.f}{2} + \overline{b-n}.a'''.\overline{e-f} + \overline{b-n}.a'''.f$
$+ \frac{\overline{b-n}.\overline{c-e}.a'''}{2}$, and so on in the other years. These several fractions being expanded, will form seven series, whose sum will be $=$ S into $\frac{\overline{r-1}.\overline{V-A}}{2r} - \frac{\overline{r-1}.\overline{BC-ABC}}{6r} - \frac{\varkappa}{3c} \times \overline{\overline{BK} -}$
$\overline{\overline{ABK} + \frac{\beta.\overline{FK-AFK}}{2b}} + \frac{m.\overline{PC-APC}}{3br} - \frac{d}{6cr} \times \overline{\overline{BT} - \overline{ABT} -}$
$\overline{\frac{m.\overline{PT-APT}}{b}} + \frac{\beta}{2b} \times \overline{\frac{FC-AFC}{3} - \overline{F} - \overline{AF}} + \frac{\overline{B-C}+\overline{AC-AB}}{2r} +$
$\frac{\varkappa.\overline{K-AK}}{2c}$.

When B is the oldest of the three lives, let x be the number of years between the age of B and of the oldest person in the table; p the number of persons living opposite the age of C; and q the same number opposite the age of A, at the end of x years; A′, A″C, and A″K, the values of annuities on those single and joint lives for x years. The given sum may be received after the first x years, on the death of A, provided C shall have died *after* B in that time, or if C shall have survived the said term of x years. Let π denote the first probability,* and $\frac{p}{c}$ will denote the second; then will the value of this part of the reversion be $=$ S . into $\overline{\pi + \frac{p}{c} \times \frac{q.\overline{r-1}.\overline{V-A^x}}{ar^{x+1}}}$, and the whole value of the reversion will be $=$ S . into $\frac{\overline{r-1}}{6r} \times \overline{2V - \overline{3A'-ABC}}$

* Found by the Table in Phil. Trans. for the year 1794, page 229.

$$-\frac{\varkappa}{3c}\times\overline{BK-ABK}+\frac{\beta\varkappa\,.\,AFK}{6bc}+\frac{m}{3br}\times\overline{PC-APC-\frac{d\,.\,\overline{1+APT}}{2c}}$$
$$+\frac{\beta}{6b}\times\overline{FC-AFC+3AF}-\frac{d}{6cr}\times\overline{BT-ABT}-\frac{\varkappa}{2c}\times A'K'$$
$$+\frac{A'C'-AB}{2r}+\pi+\frac{p}{c}\times\frac{q\,.\,\overline{r-1}\,.\,\overline{V-A^x}}{ar^{x+1}}.$$

When C is the oldest of the three lives, let φ denote the probability that C dies after B;* let z be the difference between the age of C and that of the oldest person in the table; A′, A′F′, and A′B′ the values of annuities on those single and joint lives for z years; and k the number of persons living opposite a life z years older than A; then will the value of the reversion in this case be $=$ S into $\frac{r-1}{6r}\times\overline{2V-\overline{3A'-ABC}-\frac{\varkappa}{3c}\times\overline{BK}}$ $\overline{-ABK}+\frac{\beta\varkappa}{6bc}\times AFK+\frac{m}{3br}\times\overline{PC-APC-\frac{d\,.\,\overline{1+APT}}{2c}}-$ $\frac{d}{6cr}\times\overline{BT-ABT}+\frac{\beta}{6b}\times\overline{FC-AFC+3A'F'}-\frac{\varkappa}{2c}\times$ $AK+\frac{AC-A'B'}{2r}+\frac{\varphi k\,.\,\overline{r-1}\,.\,\overline{V-A^z}}{ar^{z+1}}$.

If A be the oldest of the three lives, let b', b'', b''', *&c.* be substituted for $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, *&c.* and $\overline{a-s}$, $\overline{s-t}$, *&c.* for a', a'', *&c.* then will the value of the given sum for the 1st year be $=\frac{S}{abcr}\times\frac{\overline{a-s}\,.\,\overline{c-d}\,.\,b'}{3}+\frac{\overline{a-s}\,.\,db'}{2}$; for the 2d year $=$ $\frac{S}{abcr^2}\times\overline{\frac{\overline{s-t}\,.\,\overline{d-e}\,.\,b''}{3}+\frac{\overline{s-t}\,.\,eb''}{2}+\overline{s-t}\,.\,\overline{d-e}\,.\,b'+\overline{s-t}\,.\,eb'}$ $\overline{+\frac{\overline{c-d}\,.\,\overline{s-t}\,.\,b'}{2}}$, and so on for the other years. Hence the whole value may be found $=$ S into $\frac{\overline{r-1}}{6r}\times\overline{2V-\overline{3A-ABC}-\frac{\varkappa}{3c}}$

* By the Table in Phil. Trans. for the year 1794, page 229.

$\times \overline{AK - ABK} - \frac{\alpha x \,.\, HBK}{3ac} + \frac{\alpha}{6a} \times \overline{HC - HBC} - \frac{d}{6cr} \times \overline{\overline{AT - ABT} + \frac{s \,.\, \overline{1 + NBT}}{a} - \frac{s}{2ar} \times \frac{NC - NBC}{3} - \overline{1 + NB}} + \frac{AC - AB}{2r}$.

When the lives are equal, the value, by the two first rules, will be $= \frac{S \,.\, \overline{r-1}}{6r} \times \overline{2V - \overline{3C - CCC}} + S$ into $- \frac{x}{6c} + \overline{CK - CCK} + \frac{xx}{6cc} CKK + \frac{d}{6cr} \times \overline{CT - CCT} - \frac{dd}{6ccr} \times \overline{1 + CTT}$; and by the third rule it will be $= S$ into $\frac{r-1}{6r} \times \overline{2V - \overline{3C - CCC}} - \frac{xx}{6cc} \times CKK + \frac{d}{6cr} \times CT + \frac{d}{3cr} \times CCT + \frac{d}{2cr} - \frac{dd}{6ccr} \times \overline{1 + CTT}$. In both cases, all the fractions after the first destroy each other, so that the general rule is $= \frac{S \,.\, \overline{r-1}}{6r} \times \overline{2V - \overline{3C - CCC}}$, which may be proved, from other principles, to be the true value.

As the reversion, in this problem, consists of two parts; 1st, of the contingency of receiving the given sum on the death of A, provided B should be then dead and C living; and 2dly, of the contingency of receiving it on the death of A, provided B should be the first, C the second, and A the third, that fails, it follows, that the present solution may be obtained from those of the problem in my second Paper,* and of the 6th problem† in my last Paper on this subject. But the computations derived from the addition of those two problems would be too tedious and complicated, and therefore the preceding rules are preferable. In the particular case of the *equality* of the lives, the general rule becomes the same as above, (or $= S \times \frac{r-1}{6r} \times$

* Phil. Trans. Vol. LXXIX. page 41. † Phil. Trans. for the year 1794, page 253.

$\overline{2V - 3C - CCC})$ and consequently affords an additional proof of the truth of the investigation of the three problems.

PROBLEM VI.

To determine the value of a given sum, payable on the death of A, should his life be the *first* or *last* that shall fail, of the three lives; and should B's life, if it fail, become extinct before the life of C.

Solution.

The value of the given sum, in the 1st year, is $\frac{S}{abcr}$ into $\frac{a' . \overline{b-m} . \overline{c-d}}{2} + a'md + \frac{a' . \overline{b-m} . d}{2} + \frac{a' . \overline{c-d} . m}{2}$; in the 2d year, its value is $= \frac{S}{abcr^2}$ into $\frac{a'' . \overline{m-n} . \overline{d-e}}{2} + a''ne + \frac{a'' . \overline{m-n} . e}{2} + \frac{a'' . \overline{d-e} . n}{2} + \frac{\overline{b-m} . \overline{d-e} . a''}{2} + \frac{\overline{b-m} . \overline{c-d} . a''}{2}$; in the 3d year, $\frac{S}{abcr^3}$ into $\frac{a''' . \overline{n-o} . \overline{e-f}}{2} + a'''of + \frac{a''' . \overline{n-o} . f}{2} + \frac{a''' . \overline{e-f} . o}{2} + \frac{\overline{b-n} . \overline{e-f} . a'''}{2} + \frac{\overline{b-n} . \overline{c-e} . a'''}{2}$, and so on in the other years. The sum of these several fractions, after reducing them into their proper series, may be found $= S$ into $\frac{r-1}{2r} \times \overline{V - A - ABC} + \frac{AC}{2} - \frac{AB}{2r} + \frac{md}{2bcr} \times \overline{1 + APT} - \frac{d}{2cr} \times \overline{1 + AT + BT - ABT} - \frac{\beta x}{2bc} \times AFK + \frac{\beta}{2b} \times \overline{AF + FC - AFC}$.

When B is the oldest of the three lives, let x be the difference between his age and that of the oldest person in the table; and, as the given sum may be received after the necessary extinction of his life, either on the event of C's having died *after* him in x

years, and A's dying in the $\overline{x+1}$, $\overline{x+2}$, *&c.* years, or on the event of C's having lived x years, and A's dying in the $\overline{x+1}$, $\overline{x+2}$, *&c.* years, it is obvious that the preceding rule in this case will not express the whole value of the reversion. Retaining the same symbols as in the first case of Prob. V. the value depending on the former of those two events will be $= S \times \frac{\pi \cdot q \cdot \overline{r-1} \cdot \overline{V-A^x}}{a r^x + 1}$, and on the latter of them it will be $=$

$$\frac{AC^x}{2r^x+1} - \frac{d}{2cr^x+1} \times AT^x + \frac{pq \cdot \overline{r-1} \cdot \overline{V-A^x}}{acr^x+1} - \frac{pq \cdot \overline{r-1} \cdot \overline{V-AC^x}}{2acr^x+1};$$

therefore, the whole value of the reversion will be $=$ S into

$$\frac{r-1}{2r} \times \overline{V - A' - ABC} + \frac{AC}{2} - \frac{AB}{2r} + \frac{md}{2bcr} \times \overline{1 + APT} - \frac{d}{2cr} \times \overline{1 + AT + BT - ABT} - \frac{\beta\varkappa}{2bc} \times AFK + \frac{\beta}{2b} \times \overline{AF + FC - AFC} - \frac{pq \cdot \overline{r-1} \cdot \overline{V-AC^x}}{2acr^x+1} + \overline{\pi + \frac{p}{c}} \times \frac{q \cdot \overline{r-1} \cdot \overline{V-A^x}}{ar^x+1}.$$

When C is the oldest of the three lives, let φ denote the probability that C dies after B;* and let z be the difference between the age of C and that of the oldest person in the table; A', A'B', and A'F', the values of annuities on those single and joint lives for z years; and q the number of persons living at the age of A after z years; then will the value in this case be $=$ S into

$$\frac{r-1}{2r} \times \overline{V - A' - ABC} + \frac{AC}{2} - \frac{A'B'}{2r} + \frac{md}{2bcr} \times \overline{1 + APT} - \frac{d}{2cr} \times \overline{1 + AT + BT - ABT} - \frac{\beta\varkappa}{2bc} \times AFK + \frac{\beta}{2b} \times \overline{A'F' + FC - AFC} + \frac{\varphi q \cdot \overline{r-1} \cdot \overline{V-A^z}}{ar^z+1}.$$

When A is the oldest of the three lives, let the symbols be changed in like manner as in the corresponding case in the

* Phil. Trans. for the year 1794, page 229.

preceding problem; then will the value for the 1st year be $= \frac{S}{abcr}$ into $\frac{\overline{a-s}.\overline{c-d}.b'}{2} + \overline{a-s}.\overline{b-b'}.d + \frac{\overline{a-s}.db'}{2} + \frac{\overline{a-s}.\overline{c-d}.\overline{b-b'}}{2}$; for the 2d year $= \frac{S}{abcr^2}$ into $\frac{\overline{s-t}.\overline{d-e}\ b''}{2} + \overline{s-t}.\overline{b-\overline{b'+b''}}.e + \frac{\overline{s-t}.eb''}{2} + \frac{\overline{s-t}.\overline{d-e}.\overline{b-b'+b''}}{2} + \frac{\overline{s-t}.\overline{d-e}.b'}{2} + \frac{\overline{s-t}.\overline{c-d}.b'}{2}$, and so on for the 3d, 4th, and remaining years. These several fractions may be expanded into twelve different series, whose sum may be found $=$ S into $\frac{r-1}{2r} \times \overline{V - A - ABC + \frac{AC}{2} - \frac{AB}{2r} - \frac{\varkappa . AK}{2c} + \frac{\alpha . HBC}{2a}} - \frac{d}{2cr} \times \overline{AT - ABT} + \frac{s}{2ar} \times \overline{1 + NC + NB - NBC} - \frac{ds}{2acr} \times \overline{1 + NBT}$.

If the lives be equal, the value, by the first two rules, will be $=$ S into $\frac{\overline{r-1}.\overline{V-C+CC-CCC}}{2r} + \frac{dd}{2ccr} \times \overline{1 + CTT} - \frac{d}{2cr} \times \overline{1 + 2CT - CCT} - \frac{\varkappa\varkappa . CKK}{2cc} + \frac{\varkappa}{2c} \times \overline{2CK - CCK}$; and by the third rule it will be $=$ S into $\frac{\overline{r-1}.\overline{V-C+CC-CCC}}{2r} + \frac{\varkappa}{2c}.\overline{CCK - CK} + \frac{d}{2cr} \times \overline{1 + CT} - \frac{dd}{2ccr} \times \overline{1 + CTT}$. In both cases, all the fractions after the first destroy each other; so that the general rule becomes $= \frac{S.\overline{r-1}}{2r} \times \overline{V - C + CC - CCC}$, which may be proved to be the true value, from other principles.

The solution of this problem might have been derived from that of the first problem* in my third Paper, and from that of the 6th problem† in my last Paper on this subject, "by finding "the value on the death of A, should his life be the *first* that

* Phil. Trans. Vol. LXXXI. page 248.

† Phil. Trans. for the year 1794, page 253.

"failed; and also the value of the same sum, should B be the "first, C the second, and A the third, that failed:" but the foregoing rules, in being more simple, are preferable.

In the particular case of the *equality of the three lives*, the value, by these problems, will be $= S$ into $\frac{r-1}{3r} \times \overline{V - CCC} + \frac{r-1}{6r} \times \overline{V - 3C + 3CC - CCC} = \frac{S \,.\, \overline{r-1}}{2r} \times \overline{V - C + CC - CCC}$, as before. *Q. E. D.*

PROBLEM VII.

To determine the value of a given sum, payable on the death of A, B, and C, provided C shall die after one life in particular, (A).

Solution.

In the 1st year, the value of the given sum will be $= \frac{S}{2abcr} \times \overline{b-m} \,.\, \overline{c-d} \,.\, a'$; in the 2d year, it will be $= \frac{S}{abcr^2}$ into $\frac{\overline{m-n} \,.\, \overline{d-e} \,.\, a''}{2} + \overline{b-m} \,.\, \overline{d-e} \,.\, a' + \frac{\overline{c-d} \,.\, \overline{m-n} \,.\, a'}{2} + \overline{m-n} \,.\, \overline{d-e} \,.\, a' + \frac{\overline{b-m} \,.\, \overline{d-e} \,.\, a''}{2}$; in the 3d year $= \frac{S}{abcr^3}$ into $\frac{\overline{n-o} \,.\, \overline{e-f} \,.\, a'''}{2} + \overline{b-n} \,.\, \overline{e-f} \,.\, \overline{a'+a''} + \frac{\overline{c-e} \,.\, \overline{n-o} \,.\, \overline{a'+a''}}{2} + \overline{n-o} \,.\, \overline{e-f} \,.\, \overline{a'+a''} + \frac{\overline{b-n} \,.\, \overline{e-f} \,.\, a'''}{2}$, and so on in the other years; whence the whole value may at last be found $= S$ into $\frac{\overline{r-1}}{2r} \times \overline{BC - B - ABC + \frac{BC}{2} - \frac{AB}{2r} + \frac{m}{2br} \times \overline{1 + AP - \frac{d \,.\, \overline{1+APT}}{c}} - \frac{\kappa}{2c} \times \overline{BK - ABK}} + R$, (R denoting the value of S, by the 3d problem in my first Paper, on the contingency of C's dying *after* A).* This general rule gives the true value of the reversion, *when B is the oldest of the three lives.* But, *when C is the oldest of the three lives,* the general rule will be = S into $\frac{\overline{r-1}}{2r} \times \overline{BC - B'} \overline{- ABC} + \frac{BC}{2} - \frac{A'B'}{2r} + \frac{m}{2br} \times \overline{1 + A'P' - \frac{d . \overline{1+APT}}{c}} - \frac{x}{2c} \times \overline{BK - ABK} + R + \frac{\mu p . \overline{r-1} . \overline{V - B^x}}{b r^{x+1}}$. — x denoting the difference between the ages of C and of the oldest person in the table; p the number of persons living at the age of B after x years; B′, A′B′, and A′P′, the values of annuities on those single and joint lives for x years; and μ the probability that C dies *after* A.†

When A is the oldest of the three lives, let the symbols be changed as in the solution of the preceding problem, and the whole value of the given sum will be $= \frac{S}{2b} \times \overline{\frac{b'}{r} + \frac{b''}{r^2} + \frac{b'''}{r^3}, \&c.} + \frac{S}{2bc} \times \overline{\frac{cb'}{r} + \frac{db''}{r^2} + \frac{eb'''}{r^3}, \&c.} + \frac{S}{2bcr} \times \overline{\frac{db'}{r} + \frac{e . \overline{b'+b''}}{r^2} +, \&c.} - \frac{S}{bc} \times \overline{\frac{db'}{r} + \frac{eb''}{r^2} + \frac{f . b'''}{r^3} +, \&c.} - \frac{S}{bcr} \times \overline{\frac{eb'}{r} + \frac{f . \overline{b'+b''}}{r^2} +, \&c.} - \frac{S}{2ab} \times \overline{\frac{ab'}{r} + \frac{s . b''}{r^2} + \frac{tb'''}{r^3} +, \&c.} - \frac{S}{2abc} \times \overline{\frac{scb'}{r} + \frac{dtb''}{r^2} + \frac{eub'''}{r^3} +, \&c.} - \frac{S}{2abcr} \times \overline{\frac{dtb'}{r} + \frac{eu . \overline{b'+b''}}{r^2} +, \&c.} + \frac{S}{2abc} \times \overline{\frac{ab'd}{r} + \frac{sb''e}{r^2} + \frac{tb'''f}{r^3}, \&c.} + \frac{S}{2abcr} \times \overline{\frac{seb'}{r} + \frac{tf . \overline{b'+b''}}{r^2} +, \&c.} + \frac{S}{2abc} \times \overline{\frac{sdb'}{r} + \frac{teb''}{r^2} + \frac{ufb'''}{r^3} +, \&c.} + \frac{S}{2abcr} \times \overline{\frac{teb'}{r} + \frac{uf . \overline{b'+b''}}{r^2}, \&c.} - \frac{S}{2abcr} \times \overline{\frac{dsb'}{r} + \frac{te . \overline{b+b''}}{r^2} +, \&c.} \ldots.$

* Phil. Trans. Vol. LXXVIII. page 347.

† Phil. Trans. for the year 1794, table in page 229.

After the extinction of A's life, the given sum may be received on either of three events; 1st. on the death of B in the $\overline{z+1}$, $\overline{z+2}$, &c. years, (z denoting the difference between the ages of A and of the oldest person in the table,) C having died *after* A in the first z years; 2dly, on the death of C in the $\overline{z+1}$, $\overline{z+2}$, &c. years, B having died in the first z years; 3dly, on the extinction of both the lives of B and C after the first z years. Let φ denote the probability that C dies after A in z years;* p the number of persons living opposite the age of B; and k the same number opposite the age of C at the end of z years; and the value on the two first of these contingencies will be $= \frac{S \cdot \varphi p \cdot \overline{r-1} \cdot \overline{V-B^z}}{b r^{z+1}} + \frac{S \cdot \overline{b-p} \cdot k \cdot \overline{r-1} \cdot \overline{V-C^z}}{b c r^{z+1}}$. Again, let *greek* letters be substituted for the corresponding *italic* letters in the first part, and the value on the third of those contingencies will be $= \frac{S}{b r^z} \times \overline{\frac{\beta'}{r} + \frac{\beta''}{r^2} + \frac{\beta'''}{r^3}, \&c.} - \frac{S}{b c r^z} \times \overline{\frac{\delta \beta'}{r} + \frac{\varepsilon \beta''}{r^2} + \frac{\zeta \beta'''}{r^3} +, \&c.} + \frac{S}{b c r^{z+1}} \times \overline{\frac{\delta \beta'}{r} + \frac{\varepsilon \cdot \overline{\beta' + \beta''}}{r^2}, \&c.} - \frac{S}{b c r^{z+1}} \times \overline{\frac{\varepsilon \beta'}{r} + \frac{\zeta \cdot \overline{\beta' + \beta''}}{r^2} +, \&c.}$ The 1st of these series being added to the first series in the former part of the solution, their sum will be $=$ S into $\frac{\overline{r-1} \cdot \overline{V-B}}{2r} + \frac{p \cdot \overline{r-1} \cdot \overline{V-B^z}}{2 b r^{z+1}}$; the 2d and 4th being added to the 4th and 5th, their sum will be $= - S \times \overline{C - BC}$; and the 3d series being added to the 3d series in that part of the solution, their sum will be $= \frac{S \cdot \overline{C - BC}}{r}$; so that these six series last mentioned are $= - \frac{S \cdot \overline{r-1} \cdot \overline{C-BC}}{r}$.† The whole value of

* Phil. Trans. for the year 1794, table in page 229.

† The solution of the latter part of the case in the 6th problem, in which B is the

the reversion in this case may therefore be found = S . into $\frac{r-1}{2r} \times \overline{V - B - 2C + 2BC + AC - ABC} + \frac{B'C' - AB}{2r} + \frac{\alpha}{2a} \times \overline{HB + HC - HBC} - \frac{\varkappa}{2c} \times \overline{B'K' + AK - ABK} + \overline{\varphi + \frac{1}{2}} \times \frac{p . \overline{r-1} . \overline{V - B^{z}}}{b r^{z+1}} + \frac{\overline{b-p} . k . \overline{r-1} . \overline{V - C^{z}}}{b c r^{z+1}}$; B′C′, and B′K′, denoting the values of annuities on those joint lives for *z* years.

When the three lives are of equal age, the value by the first two rules will be = S into $\frac{r-1}{2r} \times \overline{V - 3C + 3CC - CCC} + \frac{d}{2cr} \times \overline{1 + CT} - \frac{dd}{2ccr} \times \overline{1 + CTT} - \frac{\varkappa}{2c} \times \overline{CK - CCK}$, in which the last three fractions destroy each other; and by the last rule the value will be $= \frac{S . \overline{r-1}}{2r} \times \overline{V - 3C + 3CC - CCC}$; that is, in both cases, "half the reversion after the extinction of "the three lives," which from self-evident principles is known to be the true value.

The solution of this problem may also be derived from those of the 3d problem in my first Paper,* and of the 1st problem in my last Paper,† "by deducting the value of an *estate* after the "death of C, provided that should happen after the death of A, "from the value of an annuity on the life of B after C, pro-"vided C should die before A." Thus, in the case of *equal* lives, the value by the first of these problems being $\frac{V - 2C + CC}{2}$, and the value by the second being $= \frac{C - CC}{2} - \frac{CC - CCC}{2}$, their difference, or $\frac{V - 3C + 3CC - CCC}{2}$, is the number of years pur-

eldest of the three lives, has been investigated much in the same manner with the present case; but the operation was omitted merely for the sake of conciseness.

* Phil. Trans. Vol. LXXVIII. page 347.

† Phil. Trans. for the year 1794, page 235.

chase required, and consequently the value of a given sum is $= S \times \frac{r-1}{2r} \times \overline{V - 3C + 3CC - CCC}$, as before. But the rules derived from the foregoing solution are in general more simple than those derived from the two problems just mentioned, and are therefore to be preferred to them.

The foregoing problems, together with those which have been investigated in my former papers, comprehend, as far as I can perceive, all the different cases of survivorship between three lives. The great number of contingencies on which these reversions depend, must necessarily render the solutions intricate, and consequently the general rules complicated and laborious. It would not, however, be a difficult task to abridge these rules very considerably, without destroying their accuracy in any great degree; but this would be foreign to my purpose in these papers, which has uniformly been confined to the investigation of the *correct* values of the different reversions. Nor do I think that such an abridgement is necessary, as the operations of even the longest of the present rules, may be completed in very nearly as short a time as the inaccurate approximations which have hitherto been employed for the same purpose.

It may not be improper to observe, that the solutions in these papers are not only the first which have ever been deduced, in the case of two and three lives, from just principles and the real probabilities of life; but that, as to many of the problems, not even an attempt has ever been made to *approximate* to the value of the reversion.

Being now possessed of correct solutions of all the cases in which two and three lives are involved in the survivorship, we are possessed of all that is really useful, and therefore I feel the greater satisfaction in closing my inquiries on this subject. For, in regard to contingencies depending on four or more lives, the cases are not only much too numerous and intricate to admit of a solution, but they occur so seldom in practice, as to render the entire investigation of them, were it even possible, a matter of little or no importance.

In my last Paper, printed in the Phil. Trans. for the year 1794, page 257, last line but one,

for $-\frac{BC-B'C'}{2r}$, read $-\frac{pk.\overline{r-1}.\overline{V-BC}^{x}}{2bcr^{x+1}}$; whence the general rule in the following page becomes $= S$ into $\frac{r-1}{6r}\times\overline{V-3C'-ABC}+\frac{BC}{2}-\frac{AC}{2r}-\frac{\beta}{2b}\times\overline{\frac{AF-AFC}{3}+\frac{\alpha.HFC}{3a}}+\frac{\alpha}{2a}\times\overline{\frac{2.\overline{HB}-HBC}{3}+HC}+\frac{s}{6ar}\times\overline{\overline{NB}-\overline{NBC}+\frac{m.\overline{1+N}PC}{b}}-\frac{m}{2br}\times\overline{\overline{1+PC}+\frac{2.\overline{AP}-APC}{3}}+\overline{\pi+\frac{p}{b}}\times\frac{k.\overline{r-1}.\overline{V-C}^{x}}{cr^{x+1}}-\frac{pk.\overline{r-1}.\overline{V-BC}^{x}}{2bcr^{x+1}}$.